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*A NOTE ON THE PROBLEM OF INTRODUCING SPHERICITY CORRECTION TO CONSERVATIVE SCATTERING STELLER ATMOSPHERE MODEL

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ABSTRACT. The relative magnitude of the correction to be applied for the curvature in the study of the radiative transfer problem in the conservative scattering stellar atmosphere has been calculated. Wick-Chandrasekhar method of solution for the integro-differential equation of transfer has been employed in the first approximation. The results are shown in a tabular as well as graphical form.

The problem of transfer of radiation in the case of extended atmospheres of stars in which the curvatures of the layers of atmospheres has significant effect on the transfer problem, has been considered by Kosirev (1934) and Chandrasekhar (1934). On the other hand, it is quite well known in the theory of radiative transfer that for most stars such as the sun, the atmosphere can be considered to be plane parallel. In a recent paper, Barbier (1956) discussed the problem for near solar case and established that it is not necessary to consider the atmosphere as spherically symmetrical. The purpose of the present note is to examine the magnitude of correction involved in passing from plane parallel to the spherically symmetrical case for different values of optical thickness of the stellar atmosphere in the conservative scattering case.

In the present case, the atmosphere chosen is thin, finite and conservative scattering. The equation of transfer appropriate to the problem of spherically symmetric atmosphere is given by (Radiative Transfer—Chandrasekhar, p. 364)

$$\mu \frac{\partial I(r,\mu)}{\partial r} + \frac{p-\mu^2}{r} \frac{\partial I(r,\mu)}{\partial \mu} = -k\rho I(r,\mu) + \tfrac{1}{2} k\rho \int_{-1}^{+1} I(r,\mu')d\mu' \quad \dots \quad (1)$$

and appropriate equation for the plane case is given by

$$\mu \frac{dI}{dr}(r,\mu) = -k\rho I(r,\mu) + \tfrac{1}{2} k\rho \int_{-1}^{+1} I(r,\mu')d\mu' \quad \dots \quad (2)$$

Here $I(r,\mu)$ is the specific intensity of radiation at a distance r from the centre of the star in a direction μ where $\mu = \cos \nu$, ν being the angle between the direction of the ray and the outward drawn normal to the atmosphere. Both of these equations are solved by the method of Wick-Chandrasekhar in the first approximation. Thus Eq. (1) is transformed into the following group of equations (Radiative Transfer—Chandrasekhar, p. 366) :

$$\left.\begin{aligned} \frac{d}{dr}\Sigma a_i\mu_i I_i + \frac{2}{r}\Sigma a_i\mu_i I_i = 0 \\ \frac{d}{dr}\Sigma a_i\mu_i^2 I_i + \frac{1}{r}\Sigma a_i(3\mu_i^2-1)I_i = -k\rho\Sigma a_i\mu_i I_i \end{aligned}\right\} \quad \ldots (3)$$

And for the plane parallel case, the Eq. (2) can be written out formally by dropping out the second term on the right hand side of (1) and that in the Eq. (129) of Radiative Transfer, p. 366.

$$\left.\begin{aligned} \frac{d}{dr}\Sigma a_i\mu_i I_i = 0 \\ \frac{d}{dr}\Sigma a_i\mu_i^2 I_i = -k\rho\Sigma a_i\mu_i I_i \end{aligned}\right\} \quad \ldots (4)$$

In the case of the first approximation $a_{+1} = a_{-1} = 1$ and $\mu_{+1} = \frac{1}{\sqrt{3}} = -\mu_{-1}$. The boundary conditions under which the equation is to be solved are the following :

(1) At the inner boundary of the atmosphere denoted by $r = r_0$, the outward intensity I_{+1} is very strong in comparison with I_{-1}, the inward intensity.

(2) At the outer boundary of the atmosphere denoted by $r = R$, there is no incident radiation from outside (I_{-1} is zero).

The solution of the first of Eqs. (3) yields

$$\frac{1}{2}F = \Sigma a_i\mu_i I_i = \frac{1}{2}\cdot\frac{F_0}{r^2}, \text{ where } F_0 \text{ is a constant.}$$

Hence

$$I_{+1}-I_{-1} = \frac{\sqrt{3}}{2}\cdot\frac{F_0}{r^2} \quad (5)$$

The second of Eq. (3) solves as

$$\frac{1}{3}\frac{d}{dr}(I_{+1}+I_{-1}) = -\frac{k\rho}{\sqrt{3}}(I_{+1}-I_{-1}) = \frac{k\rho}{2}\cdot\frac{F_0}{r^2}$$

$$(I_{+1}+I_{-1}) = \frac{3}{2}F_0\int_r^{\infty}\frac{k\rho dr}{r^2}+A \quad (6)$$

where A is a constant, and this can be determined by the boundary condition (1)

$$A = \frac{\sqrt{3}}{2} \cdot \frac{F_0}{r_0^2} - \frac{3}{2} F_0 \int_{r_0}^{R} \frac{k\rho dr}{r^2} \qquad \dots (7)$$

$$\therefore \qquad I_{+1}+I_{-1} = \frac{\sqrt{3}}{2} \cdot \frac{F_0}{r_0^2} - \frac{3}{2} F_0 \int_{r_0}^{R} \frac{k\rho dr}{r^2} + \frac{3}{2} F_0 \int_{r}^{R} \frac{k\rho dr}{r^2} \qquad \dots (8)$$

At $r = R$, the outer boundary of the atmosphere

$$(I_{+1})_{r=R} = \frac{\sqrt{3}}{2} \frac{F_0}{r_0^2} - \frac{3}{2} F_0 \int_{r_0}^{R} \frac{k\rho}{r^2} dr \qquad \dots (9)$$

In the plane case, we can proceed in the similar way. The solution of the first of the Eq. (4)

$$\sum a_i \mu_i I_i = \frac{1}{2} F = \frac{1}{2} \frac{F_0}{r_0^2} \qquad \dots (10)$$

where F_0 and r_0 have the same meaning as before. The flux is constant in the plane parallel case

$$I_{+1}-I_{-1} = \frac{\sqrt{3}}{2} \frac{F_0}{r_0^2} \qquad (11)$$

The second of Eqs. (4) yields

$$I_{+1}+I_{-1} = \frac{3}{2} \frac{F_0}{r_0^2} \int_{r}^{R} k\rho dr + B \qquad (12)$$

Using the boundary condition (1)

$$B = \frac{\sqrt{3}}{2} \frac{F_0}{r_0^2} - \frac{3}{2} \frac{F_0}{r_0^2} \int_{r_0}^{R} k\rho dr \qquad (13)$$

$$I_{+1}+I_{-1} = \frac{\sqrt{3}}{2} \frac{F_0}{r_0^2} - \frac{3}{2} \frac{F_0}{r_0^2} \int_{r_0}^{R} k\rho dr + \frac{3}{2} \int_{r}^{R} k\rho dr \qquad (14)$$

Then from the boundary condition (2), the incident radiation from outside at the outer boundary of the atmosphere may be taken to be zero

$$(I_{+1})_{r=R} = \frac{\sqrt{3}}{2} \frac{F_0}{r_0^2} - \frac{3}{2} \frac{F_0}{r_0^2} \int_{r_0}^{R} k\rho dr \qquad \dots (15)$$

In the astrophysical context, $k\rho$ is generally taken to vary as an inverse power of r (greater than unity)

$$\therefore \qquad k\rho = Cr^{-n}, \text{ where } C \text{ is a constant} \qquad \dots (16)$$

The optical thickness τ, measured from $r = R$ inwards is given by

$$\int_r^R k\rho dr = \frac{C}{n-1}\left[\frac{1}{r^{n-1}} - \frac{1}{R^{n-1}}\right] \qquad \dots (17)$$

$$\therefore \qquad \tau_1 = \frac{C}{n-1}\left[\frac{1}{r_0^{n-1}} - \frac{1}{R^{n-1}}\right] \qquad \dots (18)$$

i.e., $\tau = \tau_1$ denotes the photospheric surface at $r = r_0$. When $n = 2$, the emergent intensity $(I_{+1})_R$ for the spherical and the plane case is given by the two Eqns. (19) and (20).

For the spherical case

$$\frac{(I_{+1})_{\tau=0}}{\frac{\sqrt{3}}{2} F_0/r_0^2} = (1-\sqrt{3}\tau_1) + \frac{\tau_1}{\sqrt{3}}\left[2 - \frac{r_0}{R} - \frac{r_0^2}{R^2}\right] \qquad \dots (19)$$

and for the plane case

$$\frac{(I_{+1})_{\tau=0}}{\frac{\sqrt{3}}{2} F_0/r_0^2} = (1-\sqrt{3}\tau_1) \qquad \dots (20)$$

$$\therefore \qquad \frac{(I_{+1})_{\tau=0} \text{ (spherical)}}{(I_{+1})_{\tau=0} \text{ (plane)}} = 1 + \left[\frac{\tau_1}{\sqrt{3}(1-\sqrt{3}\tau_1)}\left(2 - \frac{r_0}{R} - \frac{r_0^2}{R^2}\right)\right] \dots (21)$$

The term within box bracket is obviously the term which arise from the curvature of the atmosphere and can be called sphericity correction in the present approximation for $n = 2$. The value of correction for different values of optical thickness τ_1 corresponding to several values $\frac{R-r_0}{r_0}$ are shown below in Table I. The results are shown in Fig. 1.

In the similar way, it can be shown that for $n = 3$, the equation corresponding to (21) turns out to be

$$\frac{(I_{+1})_{\tau=0}\text{(spherical)}}{(I_{+1})_{\tau=0}\text{(plane)}} = 1 + \left[\frac{\sqrt{3}\tau_1}{2(1-\sqrt{3}\tau_1)}\left(1 - \frac{r_0^2}{R^2}\right)\right] \qquad \dots (22)$$

The values of the correction terms within box bracket in the present approximation for $n = 3$ are noted below for different values of τ_1 and $\frac{R-r_0}{r_0}$ in Table II.

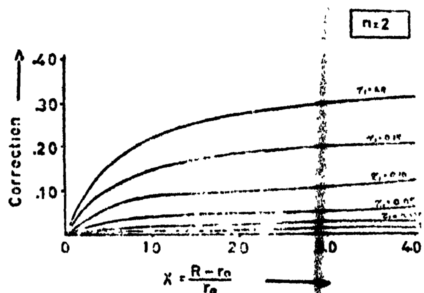

Fig. 1

TABLE I

$\tau_1 \backslash \frac{R-r_0}{r_0}$	0.25	0.50	1.00	2.00	4.00
0	0	0	0	0	0
0.01	0.003	0.003	0.007	0.009	0.010
0.025	0.008	0.013	0.019	0.023	0.026
0.050	0.018	0.028	0.037	0.047	0.056
0.10	0.039	0.062	0.087	0.098	0.123
0.15	0.066	0.104	0.146	0.182	0.206
0.20	0.099	0.157	0.221	0.275	0.311

TABLE II

$\tau_1 \backslash \frac{R-r_0}{r_0}$	0.25	0.50	1.00	2.00	4.00
0	0	0	0	0	0
0.01	0.003	0.004	0.006	0.008	0.008
0.025	0.008	0.011	0.017	0.020	0.021
0.05	0.017	0.025	0.033	0.040	0.046
0.10	0.038	0.055	0.078	0.084	0.101
0.15	0.064	0.093	0.131	0.156	0.168
0.20	0.095	0.140	0.199	0.236	0.254

The results are shown in Fig. 2.

From the graphs it is clear that there is no common region of applicability for the equation of transfer in the plane-parallel and the spherically symmetric

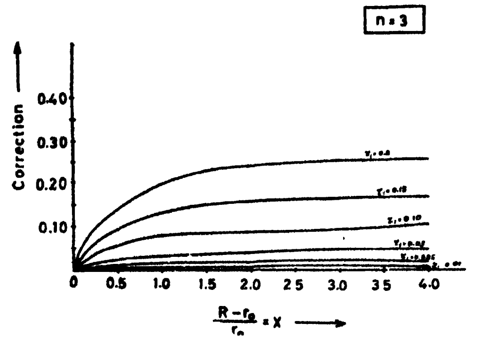

Fig. 2

case. As is physically expected, for any particular τ_1, the smaller value of $\frac{R-r_0}{r_0}$, the smaller will be the magnitude of the correction term; and for any value of $\frac{R-r_0}{r_0}$, the smaller the value of τ_1, the less significant will be the correction term.

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